

# SUPERHIGHNESS

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**ABSTRACT.** We prove that superhigh sets can be jump traceable, answering a question of Cole and Simpson. On the other hand, we show that such sets cannot be weakly 2-random. We also study the class  $\text{superhigh}^\diamond$ , and show that it contains some, but not all, of the noncomputable  $K$ -trivial sets.

## 1. INTRODUCTION

An important non-computable set of integers in computability theory is  $\emptyset'$ , the halting problem for Turing machines. Over the last half century many interesting results have been obtained about ways in which a problem can be almost as hard as  $\emptyset'$ . The *superhigh* sets are the sets  $A$  such that

$$A' \geq_{tt} \emptyset'',$$

i.e., the halting problem relative to  $A$  computes  $\emptyset''$  using a truth-table reduction. The name comes from comparison with the *high* sets, where instead arbitrary Turing reductions are allowed ( $A' \geq_T \emptyset''$ ). Superhighness for computably enumerable (c.e.) sets was introduced by Mohrherr [M]. She proved that the superhigh c.e. degrees sit properly between the high and Turing complete ( $A \geq_T \emptyset'$ ) ones.

Most questions one can ask on superhighness are currently open. For instance, Martin [M] (1966) famously proved that a degree is high iff it can compute a function dominating all computable functions, but it is not known whether superhighness can be characterized in terms of domination. Cooper [C] showed that there is a high minimal Turing degree, but we do not know whether a superhigh set can be of minimal Turing degree. We hope the present paper lays the groundwork for a future understanding of these problems.

We prove that a superhigh set can be jump traceable. Let  $\text{superhigh}^\diamond$  be the class of c.e. sets Turing below all Martin-Löf random (ML-random) superhigh sets (see [N1, Section 8.5]). We show that this class contains a promptly simple set, and is a proper subclass of the c.e.  $K$ -trivial sets. This class was recently shown to coincide with the strongly jump traceable c.e. sets, improving our result [N2].

**Definition 1.1.** Let  $\{\Phi_n^X\}_{n \in \mathbb{N}}$  denote a standard list of all functions partial computable in  $X$ , and let  $W_n^X$  denote the domain of  $\Phi_n^X$ . We write  $J^X(n)$  for  $\Phi_n^X(n)$ , and  $J^\sigma(n)$  for  $\Phi_n^\sigma(n)$  where  $\sigma$  is a string. Thus  $X' = \{e: J^B(e) \downarrow\}$  represents the halting problem relative to  $X$ .

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$X$  is jump-traceable by  $Y$  (written  $X \leq_{JT} Y$ ) if there exist computable functions  $f(n)$  and  $g(n)$  such that for all  $n$ , if  $J^X(n)$  is defined ( $J^X(n) \downarrow$ ) then  $J^X(n) \in W_{f(n)}^Y$  and for all  $n$ ,  $W_{f(n)}^Y$  is finite of cardinality  $\leq g(n)$ .

The relation  $\leq_{JT}$  is transitive and indeed a weak reducibility [N1, 8.4.14]. Further information on weak reducibilities, and jump traceability, may be found in the recent book by Nies [N1], especially in Sections 5.6 and 8.6, and 8.4, respectively.

**Definition 1.2.**  $A$  is JT-hard if  $\emptyset'$  is jump traceable by  $A$ . Let  $\text{Shigh} = \{Y : Y' \geq_{tt} \emptyset''\}$  be the class of superhigh sets.

**Theorem 1.3.** Consider the following five properties of a set  $A$ .

- (1)  $A$  is Turing complete;
- (2)  $A$  is almost everywhere dominating;
- (3)  $A$  is JT-hard;
- (4)  $A$  is superhigh;
- (5)  $A$  is high.

We have  $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ , all implications being strict.

*Proof.* Implications:  $(1) \Rightarrow (2)$ : Dobrinen and Simpson [DS].  $(2) \Rightarrow (3)$ : Simpson [S] Lemma 8.4.  $(3) \Rightarrow (4)$ : Simpson [S] Lemma 8.6.  $(4) \Rightarrow (5)$ : Trivial, since each truth-table reduction is a Turing reduction.

Non-implications:  $(2) \not\Rightarrow (1)$  was proved by Cholak, Greenberg, and Miller [CGM].  $(3) \not\Rightarrow (2)$ : By Cole and Simpson [CS], (3) coincides with (4) on the  $\Delta_2^0$  sets. But there is a superhigh degree that does not satisfy (2): one can use Jockusch-Shore Jump Inversion for a super-low but not  $K$ -trivial set, which exists by the closure of the  $K$ -trivials under join and the existence of a pair of super-low degrees joining to  $\emptyset'$ .  $(4) \not\Rightarrow (3)$ : We prove in Theorem 2.1 below that there is a jump traceable superhigh degree. By transitivity of  $\leq_{JT}$  and the observation that  $\emptyset' \not\leq_{JT} \emptyset$ , no jump traceable degree is JT-hard.  $(5) \not\Rightarrow (4)$ : Binns, Kjos-Hanssen, Lerman, and Solomon [BKHLS] proved this using a syntactic analysis combined with a result of Schwartz [S].  $\square$

Historically, the easiest separation (1)(5) is a corollary of Friedberg's Jump Inversion Theorem [F] from 1957. The separation (1)(4) follows similarly from Mohrherr's Jump Inversion Theorem for the tt-degrees [M] (1984), and the separation (4)(5) is essentially due to Schwartz [S] (1982). The classes (2) and (3) were introduced more recently, by Dobrinen and Simpson [DS] (2004) and Simpson [S] (2007).

Notion (3), JT-hardness, may not appear to be very natural. However, Cole and Simpson [CS] gave an embedding of the hyperarithmetic hierarchy  $\{0^{(\alpha)}\}_{\alpha < \omega_1^{CK}}$  into the lattice of  $\Pi_1^0$  classes under Muchnik reducibility making use of the notion of *bounded limit recursive* (BLR) functions. We will see that JT-hardness coincides with BLR-hardness.

*Notation.* We write

$$\forall n \, f(n) = \lim_s^{\text{comp}} \tilde{f}(n, s)$$

if for all  $n$ ,  $f(n) = \lim_s \tilde{f}(n, s)$ , and moreover there is a computable function  $g : \omega \rightarrow \omega$  such that for all  $n$ ,  $\{s \mid \tilde{f}(n, s) \neq \tilde{f}(n, s+1)\}$  has cardinality less than  $g(n)$ .

## 2. SUPERHIGHNESS AND JUMP TRACEABILITY

In this section we show that superhighness is compatible with the lowness property of being jump traceable, and deduce an answer to a question of Cole and Simpson.

**Theorem 2.1.** *There is a superhigh jump-traceable set.*

*Proof.* Mohrherr [M] proves a jump inversion theorem in the tt-degrees: For each set  $A$ , if  $\emptyset' \leq_{tt} A$ , then there exists a set  $B$  such that  $B' \equiv_{tt} A$ . To produce  $B$ , Mohrherr uses the same construction as in the proof of Friedberg's Jump Inversion Theorem for the Turing degrees. Namely,  $B$  is constructed by finite extensions  $B[s] \preceq B[s+1] \preceq \dots$ . Here  $B[s]$  is a finite binary string and  $\sigma \preceq \tau$  denotes that  $\sigma$  is an initial substring of  $\tau$ . At stages of the form  $s = 2e$  (even stages), one searches for an extension  $B[s+1]$  of  $B[s]$  such that  $J^{B[s+1]}(e) \downarrow$ . If none is found one lets  $B[s+1] = B[s]$ . At stages of the form  $s = 2e+1$  (odd stages) one appends the bit  $A(e)$ , i.e. one lets  $B[s+1] = B[s] \frown \langle A(e) \rangle$ . Thus two types of oracle questions are asked alternately for varying numbers  $e$ :

- (1) Does a string  $\sigma \succeq B[s]$  exist so that  $J^\sigma(e) \downarrow$ , i.e.  $B \succeq \sigma$  implies  $e \in B'$ ? (If so, let  $B[s+1]$  be the first such string that is found.)
- (2) Is  $A(e) = 1$ ?

This allows for a jump trace  $V_e$  of size at most  $4^e$ . First,  $V_0$  consists of at most one value, namely the first value  $J^\sigma(e)$  found for any  $\sigma$  extending the empty string. Next,  $V_1$  consists of the first value for  $\Phi_1^T(1)$  found for any  $\tau$  extending  $\langle 0 \rangle$ ,  $\langle 1 \rangle$ ,  $\sigma \frown \langle 0 \rangle$ ,  $\sigma \frown \langle 1 \rangle$ , respectively, in the cases:  $0 \notin A$ , and  $0 \notin B'$ ;  $0 \in A$  and  $0 \notin B'$ ;  $0 \notin A$  and  $0 \in B'$ ; and  $0 \in A$  and  $0 \in B'$ . Generally, for each  $e$  there are four possibilities: either  $e$  is in  $A$  or not, and either the extension  $\sigma$  of  $B[s]$  is found or not.  $V_e$  consists of all the possible values of  $J^B(e)$  depending on the answers to these questions.

Hence  $B$  is jump traceable, no matter what oracle  $A$  is used. Thus, letting  $A = \emptyset''$  results in a superhigh jump-traceable set  $B$ .  $\square$

**Question 2.2.** *Is there a superhigh set of minimal Turing degree?*

This question is sharp in terms of the notions (1)–(5) of Theorem 1.3: minimal Turing degrees can be high (Cooper [C]) but not JT-hard (Barnališ [B]).

Cole and Simpson [CS] introduced the following notion. Let  $A$  be a Turing oracle. A function  $f: \omega \rightarrow \omega$  is *boundedly limit computable by  $A$*  if there exist an  $A$ -computable function  $\tilde{f}: \omega \times \omega \rightarrow \omega$  such that  $\lim_s^{\text{comp}} \tilde{f}(n, s) = f(n)$ .

We write

$$\text{BLR}(A) = \{f \in \omega^\omega \mid f \text{ is boundedly limit computable by } A\}.$$

We say that  $X \leq_{BLR} Y$  if  $\text{BLR}(X) \subseteq \text{BLR}(Y)$ . In particular,  $A$  is *BLR-hard* if  $\text{BLR}(\emptyset') \subseteq \text{BLR}(A)$ .

It is easy to see that  $\leq_{BLR}$  implies  $\leq_{JT}$  (Lemma 6.8 of Cole and Simpson [CS]). The following partial converse is implicit in some recent papers as pointed out to the authors by Simpson.

**Theorem 2.3.** *Suppose that  $A \leq_{JT} B$  where  $A$  is a c.e. set and  $B$  is any set. Then  $\text{BLR}(A) \subseteq \text{BLR}(B)$ .*

*Proof.* Since  $A \leq_{JT} B$ , by Remark 8.7 of Simpson [S], the function  $h$  given by

$$h(e) = J^A(e) + 1 \text{ if } J^A(e) \downarrow, h(e) = 0 \text{ otherwise,}$$

is  $B'$ -computable, with computably bounded use of  $B'$  and unbounded use of  $B$ . This implies that  $h$  is  $\text{BLR}(B)$ . Let  $\psi^A$  be any function partial computable in  $A$ . Let  $g$  be defined by

$$g(n) = \psi^A(n) + 1 \text{ if } \psi^A(n) \downarrow, g(n) = 0 \text{ otherwise.}$$

Letting  $f$  be a computable function with  $\psi^A(n) \simeq J(f(n))$  for all  $n$ , we can use the  $B$ -computable approximation to  $h$  with a computably bounded number of changes to get such an approximation to  $g$ . So  $g$  is  $\text{BLR}(B)$ . By Lemma 2.5 of Cole and Simpson [CS], it follows that  $\text{BLR}(A) \subseteq \text{BLR}(B)$ .  $\square$

**Corollary 2.4.** *For c.e. sets  $A, B$  we have  $A \leq_{JT} B \leftrightarrow A \leq_{BLR} B$ .*

**Corollary 2.5.**  *$JT$ -hardness coincides with  $\text{BLR}$ -hardness: for all  $B$ ,  $\emptyset' \leq_{JT} B \leftrightarrow \emptyset' \leq_{BLR} B$ .*

By Corollary 2.5 and Theorem 1.3((3) $\Rightarrow$ (4)),  $\text{BLR}$ -hardness implies superhighness. Cole and Simpson asked [CS, Remark 6.21] whether conversely superhighness implies  $\text{BLR}$ -hardness. Our negative answer is immediate from Corollary 2.5 and Theorem 1.3((4) $\nRightarrow$ (3)).

### 3. SUPERHIGHNESS, RANDOMNESS, AND $K$ -TRIVIALITY

We study the class  $\text{Shigh}^\diamond$  of c.e. sets that are Turing below all ML-random superhigh sets. First we show that this class contains a promptly simple set.

For background on diagonally non-computable functions and sets of PA degree see [N1, Ch 4]. Let  $\lambda$  denote the usual fair-coin Lebesgue measure on  $2^\mathbb{N}$ ; a null class is a set  $\mathcal{S} \subseteq 2^\mathbb{N}$  with  $\lambda(\mathcal{S}) = 0$ .

**Fact 3.1** (Jockusch and Soare [JS]). *The sets of PA degree form a null class.*

*Proof.* Otherwise by the zero-one law the class is conull. So by the Lebesgue Density Theorem there is a Turing functional  $\Phi$  such that  $\Phi^X(w) \in \{0, 1\}$  if defined, and

$$\{Z : \Phi^Z \text{ is total and diagonally non-computable} \}$$

has measure at least  $3/4$ .

Let the partial computable function  $f$  be defined by:  $f(n)$  is the value  $i \in \{0, 1\}$  such that for the smallest possible stage  $s$ , we observe by stage  $s$  that  $\Phi^Z(n) = i$  for a set of  $Z$ s of measure strictly more than  $1/4$ . For each  $n$ , such an  $i$  and stage  $s$  must exist. Indeed, if for some  $n$  and both  $i \in \{0, 1\}$  there is no such  $s$ , then  $\Phi^Z(n)$  is defined for a set of  $Z$ s of measure at most  $\frac{1}{4} + \frac{1}{4} = \frac{1}{2} \not\geq \frac{3}{4}$ , which is a contradiction. Moreover, we cannot have  $f(n) = J(n)$  for any  $n$ , because this would imply that there is a set of  $Z$ s of measure strictly more than  $1/4$  for which  $\Phi^Z$  is not a total d.n.c. function. Thus  $f$  is a computable d.n.c. function, which is a contradiction.  $\square$

**Theorem 3.2** (Simpson). *The class  $\text{Shigh}$  of superhigh sets is contained in a  $\Sigma_3^0$  null class.*

*Proof.* A function  $f$  is called diagonally non-computable (d.n.c.) relative to  $\emptyset'$  if  $\forall x \neg f(x) = J^{\emptyset'}(x)$ . Let  $P$  be the  $\Pi_1^0(\emptyset')$  class of  $\{0, 1\}$ -valued functions that are d.n.c. relative to  $\emptyset'$ . By Fact 3.1 relative to  $\emptyset'$ , the class  $\{Z : \exists f \leq_T Z \oplus \emptyset' [f \in P]\}$  is null. Then, since  $\text{GL}_1$  is conull, the class

$$\mathcal{K} = \{Z : \exists f \leq_{\text{tt}} Z' [f \in P]\}$$

is also null. This class clearly contains **Shigh**.

To show that  $\mathcal{K}$  is  $\Sigma_3^0$ , fix a  $\Pi_2^0$  relation  $R \subseteq \mathbb{N}^3$  such that a string  $\sigma$  is extended by a member of  $P$  iff  $\forall u \exists v R(\sigma, u, v)$ . Let  $(\Psi_e)_{e \in \mathbb{N}}$  be an effective listing of truth-table reduction procedures. It suffices to show that  $\{Z : \Psi_e(Z') \in P\}$  is a  $\Pi_2^0$  class. To this end, note that

$$\Psi_e(Z') \in P \leftrightarrow \forall x \forall t \forall u \exists s > t \exists v R(\Psi_e^{Z'} \upharpoonright_x [s], u, v). \quad \square$$

A direct construction of a  $\Sigma_3^0$  null class containing **Shigh** appears in Nies [N2].

**Question 3.3.** *Is Shigh itself a  $\Sigma_3^0$  class?*

**Corollary 3.4.** *There is no superhigh weakly 2-random set.*

*Proof.* Let  $R$  be a weakly 2-random set. By definition,  $R$  belongs to no  $\Pi_2^0$  null class. Since a  $\Sigma_3^0$  class is a union of  $\Pi_2^0$  classes of no greater measure,  $R$  belongs to no  $\Sigma_3^0$  null class. By Theorem 3.2,  $R$  is not superhigh.  $\square$

To put Corollary 3.4 into context, recall that the 2-random set  $\Omega^{\emptyset'}$  is high, whereas no weakly 3-random set is high (see [N1, 8.5.21]).

**Corollary 3.5.** *There is a promptly simple set Turing below all superhigh ML-random sets.*

*Proof.* By a result of Hirschfeldt and Miller (see [N1, Thm. 5.3.15]), for each null  $\Sigma_3^0$  class  $\mathcal{S}$  there is a promptly simple set Turing below all ML-random sets in  $\mathcal{S}$ . Apply this to the class  $\mathcal{K}$  from the proof of Theorem 3.2.  $\square$

Next we show that  $\text{Shigh}^\diamond$  is a proper subclass of the c.e.  $K$ -trivial sets. Since some superhigh ML-random set is not above  $\emptyset'$ , each set in  $\text{Shigh}^\diamond$  is a base for ML-randomness, and therefore  $K$ -trivial (for details of this argument, see [N1, Section 5.1]). It remains to show strictness. In fact in place of the superhigh sets we can consider the possibly smaller class of sets  $Z$  such that  $G \leq_{\text{tt}} Z'$ , for some fixed set  $G \geq_{\text{tt}} \emptyset''$ . Let  $\text{MLR} = \{R : R \text{ is ML-random}\}$ .

**Theorem 3.6.** *Let  $S$  be a  $\Pi_1^0$  class such that  $\emptyset \subset S \subseteq \text{MLR}$ . Then there is a  $K$ -trivial c.e. set  $B$  such that*

$$\forall G \exists Z \in S [B \not\leq_T Z \text{ \& } G \leq_{\text{tt}} Z'].$$

**Corollary 3.7.** *There is a  $K$ -trivial c.e. set  $B$  and a superhigh ML-random set  $Z$  such that  $B \not\leq_T Z$ . Thus the class of c.e. sets Turing below all ML-random superhigh sets is a proper subclass of the c.e.  $K$ -trivials.*

*Proof of Theorem 3.6.* We assume fixed an indexing of all the  $\Pi_1^0$  classes. Given an index for a  $\Pi_1^0$  class  $P$  we have an effective approximation  $P = \bigcap_t P_t$  where  $P_t$  is a clopen set ([N1, Section 1.8]).

To achieve  $G \leq_{\text{tt}} Z'$  we use a variant of Kučera coding. Given (an index of) a  $\Pi_1^0$  class  $P$  such that  $\emptyset \subset P \subseteq \text{MLR}$ , we can effectively determine  $k \in \mathbb{N}$  such that  $2^{-k} < \lambda P$ . In fact  $k \leq K(i) + O(1) \leq 2 \log i + O(1)$  where  $i$  is the index for  $P$  (see [N1, 3.3.3]). At stage  $t$  let

$$(1) \quad y_{0,t}, y_{1,t},$$

respectively be the leftmost and rightmost strings  $y$  of length  $k$  such that  $[y] \cap P_t \neq \emptyset$ . Then  $y_0$  is left of  $y_1$  where  $y_a = \lim_t y_{a,t}$ . Note that the number of changes in these approximations is bounded by  $2^k$ .

Recall that  $(\Phi_e)_{e \in \mathbb{N}}$  is an effective listing of the Turing functionals. The following will be used in a “dynamic forcing” construction to ensure that  $B \neq \Phi_e^Z$ , and to make  $B$   $K$ -trivial. Let  $c_K$  be the standard cost function for building a  $K$ -trivial set, as defined in [N1, 5.3.2]. Thus  $c_K(x, s) = \sum_{x \leq w \leq s} 2^{-K(s(w))}$ .

**Lemma 3.8.** *Let  $Q$  be a  $\Pi_1^0$  class such that  $\emptyset \subset Q \subset \text{MLR}$ . Let  $e, m \geq 0$ . Then there is a nonempty  $\Pi_1^0$  class  $P \subset Q$  and  $x \in \mathbb{N}$  such that either*

- (a)  $\forall Z \in P \neg \Phi_e^Z(x) = 0$ , or
- (b)  $\exists s c_K(x, s) \leq 2^{-m}$  &  $\forall Z \in P_s^s \Phi_{e,s}^Z(x) = 0$ ,

where  $(P^t)_{t \in \mathbb{N}}$  is an effective sequence of (indices for)  $\Pi_1^0$  classes such that  $P = \lim_t^{\text{comp}} P^t$  with at most  $2^{m+1}$  changes.

The plan is to put  $x$  into  $B$  in case (b). The change in the approximations  $P^t$  is due to changing the candidate  $x$  when its cost becomes too large.

To prove the lemma, we give a procedure constructing the required objects.

*Procedure  $C(Q, e, m)$ . Stage  $s$ .*

- (a) Choose  $x \in \mathbb{N}^{[e]}$ ,  $x \geq s$ .
- (b) If  $c_K(x, s) \geq 2^{-m}$ , GOTO (a).
- (c) If  $\{Z \in Q_s : \neg \Phi_{e,s}^Z(x) = 0\} \neq \emptyset$  let  $P^s = \{Z \in Q : \neg \Phi_e^Z(x) = 0\}$  and GOTO (b). (In this case we keep  $x$  out of  $B$  and win.) Otherwise let  $P^s = Q$  and GOTO (d). (We will put  $x$  into  $B$  and win.)
- (d) END.

Clearly we choose a new  $x$  at most  $2^m$  times, so the number of changes of  $P^t$  is bounded by  $2^{m+1}$ .

To prove the theorem, we build at each stage  $t$  a tree of  $\Pi_1^0$  classes  $P^{\alpha,t}$ , where  $\alpha \in 2^{<\omega}$ . The number of changes of  $P^{\alpha,t}$  is bounded computably in  $\alpha$ . Stage  $t$ . Let  $P^{\emptyset,t} = S$ .

- (i) If  $P = P^{\alpha,t}$  has been defined let, for  $b \in \{0, 1\}$ ,

$$Q^{\alpha b,t} = P^{\alpha,t} \cap [y_{b,t}],$$

where the strings  $y_{b,t}$  are as in (1).

- (ii) If  $Q = Q^{\beta,t}$  is newly defined let  $e = |\beta|$ , let  $m$  equal  $n_\beta$  (the code number for  $\beta$ ) plus the number of times the index for  $Q^\beta$  has changed so far. From now on define  $P^{\beta,t}$  by the procedure  $C(Q, e, m)$  in Lemma 3.8. If it reaches (d), put  $x$  into  $B$ .

**Claim 1.** (i) For each  $\alpha$  the index  $P^{\alpha,t}$  reaches a limit  $P^\alpha$ . The number of changes is computably bounded in  $\alpha$ .

(ii) For each  $\beta$  the index  $Q^{\beta,t}$  reaches a limit  $Q^\beta$ . The number of changes is computably bounded in  $\beta$ .

The claim is verified by induction, in the form  $P^\alpha \rightarrow Q^{\alpha b} \rightarrow P^{\alpha b}$ . This yields a computable definition of the bound on the number of changes.

Clearly (i) holds when  $\alpha = \emptyset$ .

*Case  $Q^{\alpha b}$ :* we can compute by inductive hypothesis an upper bound on the index for  $P^\alpha$ , and hence an upper bound  $k_0$  on  $k$  such that  $2^{-k} < \lambda P^\alpha$ . If  $N$  bounds the number of changes for  $P^\alpha$  then  $Q^{\alpha b}$  changes at most  $N2^{k_0}$  times.

*Case  $P^\beta$ ,  $\beta \neq \emptyset$ :* Let  $M$  be the bound on the number of changes for  $Q^\beta$ . Then we always have  $m \leq M + n_\beta$  in (ii), so the number of changes for  $P^\beta$  is at most  $M2^{M+n_\beta+1}$ .

**Claim 2.** (i) Let  $e = |\beta| > 0$ . Then  $B \neq \Phi_e(Z)$  for each  $Z \in P^\beta$ .

This is clear, since eventually the procedure in Lemma 3.8 has a stable  $x$  to diagonalize with.

Given  $G$  define  $Z \leq_T \emptyset' \oplus G$  as follows. For  $e > 0$  let  $\beta = G \upharpoonright_e$ . Use  $\emptyset'$  to find the final  $P^\beta$ , and to determine  $y_{\beta,b,t}$  ( $b \in \{0,1\}$ ) for  $P = P^\beta$  as the strings in (1). Let  $y_{\beta,b} = \lim y_{\beta,b,t}$ .

Note that  $y_\gamma \prec y_\delta$  whenever  $\gamma \prec \delta$ . Define  $Z$  so that  $y_{G(e)} \prec Z$ .

For  $G \leq_{tt} Z'$  define a function  $f \leq_T Z$  such that  $G(e) = \lim_s^{\text{comp}} f(e, s)$  (i.e., a computable bounded number of changes). Given  $e$ , to define  $f \upharpoonright_e [s]$  search for  $t > s$  such that  $y_{\alpha,t} \prec Z$  for some  $\alpha$  of length  $e$ , and output  $\alpha$ . □

## REFERENCES

- [CS] Joshua A. Cole and Stephen G. Simpson, *Mass problems and hyperarithmeticality*, J. Math. Log. **7** (2007), no. 2, 125–143. MR2423947
- [B] George Barmpalias, *Tracing and domination in the Turing degrees*. To appear.
- [BKHLS] Stephen Binns, Bjørn Kjos-Hanssen, Manuel Lerman, and Reed Solomon, *On a conjecture of Dobrinen and Simpson concerning almost everywhere domination*, J. Symbolic Logic **71** (2006), no. 1, 119–136. MR2210058 (2006m:03070)
- [C] S. B. Cooper, *Minimal degrees and the jump operator*, J. Symbolic Logic **38** (1973), 249–271. MR0347572 (50 #75)
- [DS] Natasha L. Dobrinen and Stephen G. Simpson, *Almost everywhere domination*, J. Symbolic Logic **69** (2004), no. 3, 914–922. MR2078930 (2005d:03079)
- [F] Richard Friedberg, *A criterion for completeness of degrees of unsolvability*, J. Symb. Logic **22** (1957), 159–160. MR0098025 (20 #4488)
- [JS] Carl G. Jockusch Jr. and Robert I. Soare,  $\Pi_1^0$  classes and degrees of theories, Trans. Amer. Math. Soc. **173** (1972), 33–56. MR0316227 (47 #4775)
- [M] Donald A. Martin, *Classes of computably enumerable sets and degrees of unsolvability*, Z. Math. Logik Grundlagen Math. **12** (1966), 295–310. MR0224469 (37 #68)
- [S] Stephen G. Simpson, *Almost everywhere domination and superhighness*, MLQ Math. Log. Q. **53** (2007), no. 4-5, 462–482. MR2351944
- [CGM] Peter Cholak, Noam Greenberg, and Joseph S. Miller, *Uniform almost everywhere domination*, J. Symbolic Logic **71** (2006), no. 3, 1057–1072. MR2251556 (2007e:03071)
- [M] Jeanleah Mohrherr, *Density of a final segment of the truth-table degrees*, Pacific J. Math. **115** (1984), no. 2, 409–419. MR765197 (86a:03043)
- [N1] André Nies, *Computability and Randomness*, Oxford University Press, 2009.
- [N2] ———, *Superhighness and strong jump traceability*. To appear in Proc. ICALP 2009.

[S] S. Schwarz, *Index sets of computably enumerable sets, quotient lattices, and computable linear orderings*, doctoral dissertation, University of Chicago, 1982.

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